# Exact dark soliton solutions for a family of $N$ coupled nonlinear Schrödinger equations in optical fiber media

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We consider a family of $N$ coupled nonlinear Schrödinger equations which govern the simultaneous propagation of $N$ fields in the normal dispersion regime of an optical fiber with various important physical effects. The linear eigenvalue problem associated with the integrable form of all the equations is constructed with the help of the Ablowitz-Kaup-Newell-Segur method. Using the Hirota bilinear method, exact dark soliton solutions are explicitly derived.

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## I. INTRODUCTION

Hasegawa and Tappert [1,2], theoretically predicted the possibility of propagation of envelope solitons in optical fibers and it was experimentally demonstrated by Mollenauer *et al.* [3] in 1980. Since then, numerous interesting research results, both theoretical and experimental have been reported in the field of optical solitons, as they are very useful in high speed digital optical fiber communication. In the absence of optical losses, the wave dynamics of nonlinear pulse propagation in a monomode fiber is described by the famous nonlinear Schrödinger (NLS) equation [4,5] given by

$$iq_z - \frac{k''}{2} q_{tt} + \beta |q|^2 q = 0, \tag{1}$$

where $q$ represents the complex envelope amplitude, $t$ and $z$ are the time and distance along the direction of propagation, $k''$ is the second derivative of the axial wave number $k$ with respect to the angular frequency $\omega_0$ and describes group velocity dispersion, and $\beta = n_2 \omega_0 / cA_{\text{eff}}$ is the self-phase-modulation parameter, with $n_2$ the Kerr coefficient, $c$ the speed of light, and $A_{\text{eff}}$ the effective core area of the fiber.

The possibility of bright (dark) solitons in optical fibers is due to exact counterbalancing between the effects of anomalous (normal) group velocity dispersion and self-phase modulation [1–6]. Dark solitons are generally considered to be less desirable for applications in high speed communication systems because of their higher average power and resulting undesirable effects, such as excitation of stimulated Brillouin backscattering. On the other hand, bright solitons have the drawback of fully utilizing the line capacity because of the necessity of keeping relatively large separations between pulses to avoid accumulation of bit errors. Also, optical losses decrease the intensity of the pulse, along with a corresponding increase in the width. This effect is smaller in the dark optical soliton case. It was shown both numerically and analytically that the time jitter in a dark soliton is lower than in the corresponding bright soliton [7,8]. The interactive force between two dark solitons is always repulsive, unlike the bright soliton case, and decreases twice as fast as a function of the distance between the solitons. The separation increases monotonically rather than periodically as in the case of bright solitons.

For handling more channels it is necessary to propagate more than one field simultaneously. Transmission of many fields simultaneously in a fiber is called wavelength division multiplexing (WDM) (i.e., fields with slightly different frequencies). In 1974, Manakov [9] derived the coupled NLS (CNLS) equations from the NLS equation by considering the total field to be comprised of two fields (left and right polarizations). In the same work he presented the linear eigenvalue problem associated with the CNLS equations and the soliton solutions using the inverse scattering transform (IST). A Painlevé analysis of the CNLS equations was carried out by Sahadevan *et al.* [10]. Bright and dark soliton solutions using the Hirota bilinear method for the CNLS equations were presented by Radhakrishnan and Lakshmanan [11]. In [12] we generated bright soliton solutions using the Bäcklund transformation method. Very recently we constructed bright soliton solutions for the simultaneous propagation of $N$ nonlinear waves in the anomalous dispersion regime [13].

When we consider the simultaneous propagation of $N$ nonlinear waves in the normal dispersion regime of a fiber $(k'' = \beta)$, the wave dynamics of the system will be governed by $N$ CNLS equations of the form

$$q_{jz} = i\left[ -\frac{1}{2} q_{jtt} + \left( \sum_{n=1}^{N} |q_n|^2 \right) q_j \right], \quad j = 1,2,\ldots,N. \tag{2}$$

For two field propagation (i.e., $j=2$) in optical fibers the scaling factor between the cross-phase modulation and the self-phase modulation is 2/3,2, and between 2/3 and 2, corresponding to the propagation of linearly, circularly, and elliptically polarized eigenmodes respectively, in each model [14,15]. Here this ratio is equated to unity, which corresponds to to the case of elliptical bifringence [14]. In the case of many field propagation for WDM, the cross-coupling ratio between cross-phase modulation and self-phase modulation

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is 2 [4,5]. But the results of the present analysis might be useful for the perturbational analysis of related problems.

Later, it was found that the propagation of high bit rate soliton pulses is greatly influenced by higher order effects also [4,5]. Physically important higher order effects are higher order dispersion, self-steepening, stimulated inelastic scattering, and delayed nonlinear response. With the effect of all these physical processes, the wave dynamics of the optical fiber system is governed by a higher order NLS (HNLS) equation of the form [4,5]

$$iq_z-\frac{k''}{2}q_{tt}+\beta|q|^2q-\frac{ik'''}{6}q_{ttt}+i\gamma(|q|^2q)_t+i\gamma_s(|q|^2)_tq=0, \tag{3}$$

where $k'''=\partial^3k/\partial\omega^3$ at $\omega_0$ describes higher order dispersion, $\gamma=2\beta/\omega_0$ describes Kerr dispersion (also called self-steepening), and $\gamma_s$ represents the delayed nonlinear process. The imaginary part of $\gamma_s$ describes stimulated Raman scattering. We consider only the real part of $\gamma_s$. Although here we have considered only the real part, one can use the results of the analysis presented in this work as a starting point for perturbational analysis of the system equation that includes the stimulated Raman scattering term (imaginary part) also.

Kodama [16] has shown that with suitable transformation and omitting the higher order terms the HNLS equation (3) can be reduced to the Hirota equation [17]

$$iq_z-\frac{k''}{2}q_{tt}+\beta|q|^2q-\frac{ik'''}{6}q_{ttt}+i\gamma|q|^2q_t=0. \tag{4}$$

The Hirota equation was first considered by Hirota himself in [17] for the integrability condition $k''\gamma=\beta k'''$ and he derived the bright soliton solution for the anomalous dispersion regime $k''=-\beta$. Through Painlevé analysis the integrability condition $k''\gamma=\beta k'''$ was derived by Sakovich [18]. This integrability condition is valid for both normal and anomalous dispersion regimes. For the normal dispersion regime when we consider $k''=\beta$, the integrable form of $N$ coupled Hirota ($N$ CH) equations takes the form

$$\begin{aligned}q_{jz}=&i\left[-\frac{1}{2}q_{jtt}+\left(\sum_{n=1}^{N}|q_n|^2\right)q_j\right]+q_{jttt}-3\left(\sum_{n=1}^{N}|q_n|^2\right)q_{jt}\\&-3\left(\sum_{n=1}^{N}q_n^*q_{nt}\right)q_j,\quad j=1,2,\ldots,N.\end{aligned} \tag{5}$$

The case of two coupled Hirota equations was first considered by Tasgal and Potasek [19]. They constructed the Lax pair and obtained the bright soliton solutions using the IST. Radhakrishnan *et al.* [20] performed a Painlevé analysis and generated bright and dark soliton solutions for the coupled Hirota equations using the bilinear transformation method. Using the Bäcklund transformation method we generated the bright soliton solutions for the same case [12]; and also the bright soliton solutions for the $N$ CH equations [13]. Here, in Eq. (5), one can see that there are specific constraints between the coefficients of the higher order nonlinear terms. Hence the analysis on Eq. (5) may not be useful directly for physical problems. But the analysis might be useful, to help as a starting point in terms of perturbational analysis for related nonlinear fiber optical problems.

Painlevé analysis of the HNLS equation (3) has been carried out many times [21,22,18]. Through Painlevé analysis, the conditions needing to be satisfied for the integrability of Eq. (3) with all the higher order terms have been reported as [18]

$$k''\gamma=\beta k''' \quad \text{and} \quad \gamma=-2\gamma_s. \tag{6}$$

These conditions are valid for both normal and anomalous dispersion regimes of the fiber system. In the fiber system Eq. (3), if $k''$ and $\beta$ are of opposite sign, then the equation governs the pulse propagation in the anomalous dispersion regime where the bright soliton exists. Without loss of generality if we consider $k''=-\beta$ for bright soliton propagation, then the integrability conditions (6) become $\gamma=-2\gamma_s=-k'''$, and with transformations variables of

$$\begin{gathered}q=u\exp\left\{i\left[\frac{k''}{k'''}T-\frac{k''^3}{k'''^3}Z\right]\right\},\\ Z=\frac{-k'''}{6}z,\quad T=t-\frac{k''^2}{2k'''}z,\end{gathered} \tag{7}$$

Eq. (3) reduces to the following system of complex modified equation of Korteweg–de vries (KdV) type:

$$u_Z+u_{TTT}+6|u|^2u_T+3u(|u|^2)_T=0. \tag{8}$$

Sasa and Satsuma were the first to report the inverse scattering transform scheme for Eq. (8) [23]. Soliton solutions using Bäcklund transformation and the Hirota bilinear method are presented in [22]. For pulse propagation in the normal dispersion regime of the fiber system Eq. (3), $k''$ and $\beta$ should be of identical sign, which is the condition for dark solitons. For dark soliton propagation if we consider $k''=\beta$, then the integrability conditions (6) become $\gamma=-2\gamma_s=k'''$, and with the transformation of variables (7), Eq. (3) reduces to a different complex modified KdV equation:

$$u_Z-u_{TTT}+6|u|^2u_T+3u(|u|^2)_T=0. \tag{9}$$

Palacios *et al.*, derived the dark soliton solution for the HNLS equation using the coupled amplitude-phase formulation [24]. When we consider the simultaneous propagation of $N$ nonlinear waves in the fiber system with higher order effects, the HNLS equation (3) can be written in the form of $N$ coupled HNLS ($N$ CHNLS) equations as

$$iq_{jz}-\frac{k''}{2}q_{jtt}+\beta\sum_{n=1}^{N}|q_n|^2q_j-\frac{ik'''}{6}q_{jttt}+i\gamma\left(\sum_{n=1}^{N}|q_n|^2q_j\right)_t$$
$$+i\gamma_s\left(\sum_{n=1}^{N}|q_n|^2\right)_t q_j=0,\quad j=1,2,\ldots,N. \tag{10}$$

Two coupled forms of Eq. (10) have been considered in [25]. In that work the exact form of the bright and dark soliton solutions was derived using the Hirota bilinear method. Very recently, Painlevé analysis of two coupled HNLS equations was reported in [26]. We considered the integrable form of two and three coupled version of Eq. (10) and constructed the Lax pair and derived the bright soliton solutions using the Bäcklund transformation [27]. The IST scheme for $N$ coupled HNLS equations is reported in [28]. Recently we derived the bright soliton solutions for the integrable form of Eqs. (10) using the Bäcklund transformation [13].

In this paper, we consider the $N$ CNLS equations (2), $N$ CH equations (5), and $N$ CHNLS equations (10), which govern simultaneous propagation of $N$ fields in the normal dispersion regime of an optical fiber with various important physical effects. We construct the eigenvalue problem associated with all these equations with the help of the Ablowitz-Kaup-Newell-Segur (AKNS) method [29]. Using the Hirota bilinear method, we derive the exact dark one-soliton solutions.

## II. *N* CNLS EQUATIONS

In this section we consider the $N$ CNLS equations (2). The linear eigenvalue problem associated with Eq. (2) is derived using the AKNS method as

$$\frac{\partial\Psi}{\partial t}=U_1\Psi,$$
$$\Psi=(\psi_1\psi_2\psi_3\cdots\psi_{N+1})^T, \tag{11}$$

where

$$U_1=\begin{pmatrix}\zeta & iq_1 & iq_2 & \cdots & iq_N\\ -iq_1^* & -\zeta & 0 & \cdots & 0\\ -iq_2^* & 0 & -\zeta & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -iq_N^* & 0 & 0 & \cdots & -\zeta\end{pmatrix}, \tag{12}$$

and $\zeta$ is the spectral parameter. The space evolution of the the eigenfunction $\Psi$ is given by

$$\frac{\partial\Psi}{\partial z}=V_1\Psi,$$

$$V_1=i\zeta^2\begin{pmatrix}-1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}+\zeta\begin{pmatrix}0 & q_1 & q_2 & \cdots & q_N\\ -q_1^* & 0 & 0 & \cdots & 0\\ -q_2^* & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -q_N^* & 0 & 0 & \cdots & 0\end{pmatrix} \tag{13}$$

$$+\frac{i}{2}\begin{pmatrix}A & -iq_{1t} & -iq_{2t} & \cdots & -iq_{Nt}\\ -iq_{1t}^* & -|q_1|^2 & -q_2q_1^* & \cdots & -q_Nq_1^*\\ -iq_{2t}^* & -q_1q_2^* & -|q_2|^2 & \cdots & -q_Nq_2^*\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -iq_{Nt}^* & -q_1q_N^* & -q_2q_N^* & \cdots & -|q_N|^2\end{pmatrix}, \tag{14}$$

where $A=\Sigma_{n=1}^{N}|q_n|^2$. Equation (2) can be obtained from the compatibility condition $U_{1z}-V_{1t}+[U_1,V_1]=0$.

In order to construct dark soliton solutions using the Hirota method, we apply the following form of bilinear transformation to Eq. (2):

$$q_j=\frac{g_j(z,t)}{f(z,t)}, \tag{15}$$

where $g_j(z,t)$ are complex functions with respect to $z$ and $t$. Using Eq. (15), Eq. (2) can be decoupled into

$$(iD_z-D_t^2/2-\lambda_1/2)(g_jf)=0, \tag{16a}$$

$$(D_t^2+\lambda_1)ff=-2\sum_{n=1}^{N}|g_n|^2, \tag{16b}$$

in which $\lambda_1$ is a constant to be determined. To obtain the dark soliton solutions, we assume

$$g_j=g_{j0}(1+\epsilon g_{1j}),\quad f=1+\epsilon f_1. \tag{17}$$

Substituting Eq. (17) in Eq. (16) and then collecting the coefficients of $\epsilon^{(0)}$, we get

$$g_{j0}=\tau_{1j}\exp(i\psi_1), \tag{18}$$

where

$$\psi_1=\kappa_1t-(\lambda_1-\kappa_1^2)z/2+\psi_1^{(0)}, \tag{19}$$

and $\Sigma_{n=1}^{N}|\tau_{1n}|^2=-\lambda_1/2$, in which $\kappa_1$ and $\psi_1^{(0)}$ are real constants and the $\tau_{1j}$'s are complex constants. Then from the coefficients of $\epsilon^{(1)}$ and $\epsilon^{(2)}$ we derive the solutions

$$g_j=-f_1=-\exp[m_1(t+\kappa_1 z)+\xi_1^{(0)}], \tag{20}$$

where $m_1^2=-2\lambda_1=4\Sigma_{n=1}^{N}|\tau_{1n}|^2$ and $\xi_1^{(0)}$ is a real constant. Now using the solutions of $g_{1j}, f_1$, and $\lambda_1$ in Eq. (17) and then in Eq. (15), the dark soliton solutions of Eq. (2) can be derived as

$$q_j=-\tau_{1j}\exp(i\psi_1)\tanh[m_1(t+\kappa_1 z)+\xi_1^{(0)}]. \tag{21}$$

## III. *N* CH EQUATIONS

Let us consider the $N$ CH equations (5). The linear eigenvalue problem associated with Eq. (5) is derived using the AKNS method as

$$\frac{\partial\Psi}{\partial t}=U_2\Psi, \qquad \Psi=(\psi_1\psi_2\psi_3\cdots\psi_{N+1})^T, \tag{22}$$

where

$$U_2=\begin{pmatrix} \zeta & iq_1 & iq_2 & \cdots & iq_N \\ -iq_1^* & -\zeta & 0 & \cdots & 0 \\ -iq_2^* & 0 & -\zeta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -iq_N^* & 0 & 0 & \cdots & -\zeta \end{pmatrix}. \tag{23}$$

The space evolution of the eigenfunction $\Psi$ is given by

$$\frac{\partial\Psi}{\partial z}=V_2\Psi, \tag{24}$$

$$V_2=(-4\zeta^3+i\zeta^2)\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}+(4i\zeta^2+\zeta)\begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_N \\ -q_1^* & 0 & 0 & \cdots & 0 \\ -q_2^* & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_N^* & 0 & 0 & \cdots & 0 \end{pmatrix}+\left(-2\zeta+\frac{i}{2}\right)$$

$$\times\begin{pmatrix} A & -iq_{1t} & -iq_{2t} & \cdots & -iq_{Nt} \\ -iq_{1t}^* & -|q_1|^2 & -q_2q_1^* & \cdots & -q_Nq_1^* \\ -iq_{2t}^* & -q_1q_2^* & -|q_2|^2 & \cdots & -q_Nq_2^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -iq_{Nt}^* & -q_1q_N^* & -q_2q_N^* & \cdots & -|q_N|^2 \end{pmatrix}$$

$$-i\begin{pmatrix} -i\sum_{n=1}^{N}(q_{nt}q_n^*-q_nq_{nt}^*) & -q_{1tt}+2Aq_1 & -q_{2tt}+2Aq_2 & \cdots & -q_{Ntt}+2Aq_N \\ q_{1tt}^*-2Aq_1^* & i(q_{1t}q_1^*-q_1q_{1t}^*) & i(q_{2t}q_1^*-q_2q_{1t}^*) & \cdots & i(q_{Nt}q_1^*-q_Nq_{1t}^*) \\ q_{2tt}^*-2Aq_2^* & i(q_{1t}q_2^*-q_1q_{2t}^*) & i(q_{2t}q_2^*-q_2q_{2t}^*) & \cdots & i(q_{Nt}q_2^*-q_Nq_{2t}^*) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{Ntt}^*-2Aq_N^* & i(q_{1t}q_N^*-q_1q_{Nt}^*) & i(q_{2t}q_N^*-q_2q_{Nt}^*) & \cdots & i(q_{Nt}q_N^*-q_Nq_{Nt}^*) \end{pmatrix}, \tag{25}$$

where $A=\Sigma_{n=1}^{N}|q_n|^2$. Equation (5) can be obtained from the compatibility condition $U_{2z}-V_{2t}+[U_2,V_2]=0$.

In order to construct dark soliton solutions using the Hirota method, we apply the same form of bilinear transformation (15) on Eq. (5) to get

$$(iD_z-D_t^2/2-iD_t^3-3i\lambda_2D_t-\lambda_2/2)(g_jf)=0, \tag{26a}$$

$$(D_t^2+\lambda_2)ff=-2\sum_{n=1}^{N}|g_n|^2, \tag{26b}$$

in which $\lambda_2$ is a constant to be determined. To obtain the dark soliton solutions, we assume

$$g_j=g_{j0}(1+\epsilon g_{1j}),\quad f=1+\epsilon f_1, \tag{27}$$

Substituting Eq. (27) in Eq. (26) and then collecting the coefficients of $\epsilon^{(0)}$, we get

$$g_{j0}=\tau_{2j}\exp(i\psi_2), \tag{28}$$

where

$$\psi_2=\kappa_2t-[(\lambda_2-\kappa_2^2)/2+\kappa_2(\kappa_2^2-3\lambda_2)]z+\psi_2^{(0)}, \tag{29}$$

and $\Sigma_{n=1}^{N}|\tau_{2n}|^2=-\lambda_2/2$, in which $\kappa_2$ and $\psi_2^{(0)}$ are real constants and the $\tau_{2j}$'s are complex constants. Then from the coefficients of $\epsilon^{(1)}$ and $\epsilon^{(2)}$ we derive the solutions

$$g_j=-f_1=-\exp\{m_2t-[(3\kappa_2^2-3\lambda_2-\kappa_2)m_2-m_2^3]z+\xi_2^{(0)}\}, \tag{30}$$

where $m_2^2=-2\lambda_2=4\Sigma_{n=1}^{N}|\tau_{2n}|^2$ and $\xi_2^{(0)}$ is a real constant. Now using the solutions of $g_{1j},f_1$, and $\lambda_2$ in Eq. (27) and then in Eq. (15), the dark soliton solutions of Eq.(5) can be derived as

$$q_j=-\tau_{2j}\exp(i\psi_2)\tanh\{m_2t-[(3\kappa_2^2-3\lambda_2-\kappa_2)m_2-m_2^3]z+\xi_2^{(0)}\}. \tag{31}$$

## IV. *N* CHNLS EQUATIONS

In this section we construct the linear eigenvalue problem using the AKNS method and derive the dark soliton solutions using the Hirota bilinear method for the $N$ CHNLS equations (10). For $N$ field propagation in the normal dispersion regime we consider the integrability conditions $k''=\beta$ and $\gamma=-2\gamma_s=k'''$ [26] and using the transformations

$$q_j=u_j\exp\left\{i\left[\frac{k''}{k'''}T-\frac{k''^3}{k'''^3}Z\right]\right\},Z=\frac{-k'''}{6}z,\quad T=t-\frac{k''^2}{2k'''}z, \tag{32}$$

Equation (10) reduces to $N$ coupled complex modified KdV equations,

$$u_{jZ}-u_{jTTT}+6\sum_{n=1}^{N}|u_n|^2u_{jT}+3u_j\left(\sum_{n=1}^{N}|u_n|^2\right)_T=0. \tag{33}$$

The Lax pair for the $N$ coupled complex modified KdV equations (33) is derived using the AKNS method as

$$\frac{\partial\Psi}{\partial T}=U_3\Psi,\quad \frac{\partial\Psi}{\partial Z}=V_3\Psi,\quad \Psi=(\psi_1\psi_2\psi_3\cdots\psi_{2N+1})^T, \tag{34}$$

where

$$U_3=\begin{pmatrix} \zeta & 0 & \cdots & 0 & 0 & 0 & 0 & iu_N\\ 0 & \zeta & \cdots & 0 & 0 & 0 & 0 & iu_N^*\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \zeta & 0 & 0 & 0 & iu_2\\ 0 & 0 & \cdots & 0 & \zeta & 0 & 0 & iu_2^*\\ 0 & 0 & \cdots & 0 & 0 & \zeta & 0 & iu_1\\ 0 & 0 & \cdots & 0 & 0 & 0 & \zeta & iu_1^*\\ -iu_N^* & -iu_N & \cdots & -iu_2^* & -iu_2 & -iu_1^* & -iu_1 & -\zeta \end{pmatrix}, \tag{35}$$

$$V_3=4\zeta^3\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \end{pmatrix}+4i\zeta^2\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_N\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_N^*\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2^*\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1^*\\ -u_N^* & -u_N & \cdots & -u_2^* & -u_2 & -u_1^* & -u_1 & 0 \end{pmatrix}$$

$$
-2\zeta\begin{pmatrix}
|u_N|^2 & u_N^2 & \cdots & u_2^* u_N & u_2 u_N & u_1^* u_N & u_1 u_N & -iu_{NT} \\
u_N^{*2} & |u_N|^2 & \cdots & u_2^* u_N^* & u_2 u_N^* & u_1^* u_N^* & u_1 u_N^* & -iu_{NT}^* \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u_2 u_N^* & u_2 u_N & \cdots & |u_2|^2 & u_2^2 & u_2 u_1^* & u_2 u_1 & -iu_{2T} \\
u_2^* u_N^* & u_2^* u_N & \cdots & u_2^{*2} & |u_2|^2 & u_2^* u_1^* & u_2^* u_1 & -iu_{2T}^* \\
u_1 u_N^* & u_1 u_N & \cdots & u_1 u_2^* & u_1 u_2 & |u_1|^2 & u_1^2 & -iu_{1T} \\
u_1^* u_N^* & u_1^* u_N & \cdots & u_1^* u_2^* & u_1^* u_2 & u_1^{*2} & |u_1|^2 & -iu_{1T}^* \\
-iu_{NT}^* & -iu_{NT} & \cdots & -iu_{2T}^* & -iu_{2T} & -iu_{1T}^* & -iu_{1T} & -2B
\end{pmatrix}
$$

$$
+\begin{pmatrix}
u_{NT}^* u_N - u_N^* u_{NT} & 0 & \cdots & u_{2T}^* u_N - u_2^* u_{NT} & u_{2T} u_N - u_2 u_{NT} & u_{1T}^* u_N - u_1^* u_{NT} & u_{1T} u_N - u_1 u_{NT} & -4iBu_N + iu_{NTT} \\
0 & u_N^* u_{NT} - u_{NT}^* u_N & \cdots & u_{2T}^* u_N^* - u_2^* u_{NT}^* & u_{2T} u_N^* - u_2 u_{NT}^* & u_{1T}^* u_N^* - u_1^* u_{NT}^* & u_{1T} u_N^* - u_1 u_{NT}^* & -4iBu_N^* + iu_{NTT}^* \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u_2 u_{NT}^* - u_{2T} u_N^* & u_2 u_{NT} - u_{2T} u_N & \cdots & u_{2T}^* u_2 - u_2^* u_{2T} & 0 & u_2 u_{1T}^* - u_{2T} u_1^* & u_2 u_{1T} - u_{2T} u_1 & -4iBu_2 + iu_{2TT} \\
u_2^* u_{NT}^* - u_{2T}^* u_N^* & u_2^* u_{NT} - u_{2T}^* u_N & \cdots & 0 & u_2^* u_{2T} - u_{2T}^* u_2 & u_2^* u_{1T}^* - u_{2T}^* u_1^* & u_2^* u_{1T} - u_{2T}^* u_1 & -4iBu_2^* + iu_{2TT}^* \\
u_1 u_{NT}^* - u_{1T} u_N^* & u_1 u_{NT} - u_{1T} u_N & \cdots & u_1 u_{2T}^* - u_{1T} u_2^* & u_1 u_{2T} - u_{1T} u_2 & u_{1T}^* u_1 - u_1^* u_{1T} & 0 & -4iBu_1 + iu_{1TT} \\
u_1^* u_{NT}^* - u_{1T}^* u_N^* & u_1^* u_{NT} - u_{1T}^* u_N & \cdots & u_1^* u_{2T}^* - u_{1T}^* u_2^* & u_1^* u_{2T} - u_{1T}^* u_2 & 0 & u_1^* u_{1T} - u_{1T}^* u_1 & -4iBu_1^* + iu_{1TT}^* \\
4iBu_N^* - iu_{NTT}^* & 4iBu_N - iu_{NTT} & \cdots & 4iBu_2^* - iu_{2TT}^* & 4iBu_2 - iu_{2TT} & 4iBu_1^* - iu_{1TT}^* & 4iBu_1 - iu_{1TT} & 0
\end{pmatrix} \tag{36}
$$

where $B = \Sigma_{n=1}^{N} |u_n|^2$. Equation (33) can be obtained from the compatibility condition $U_{3Z} - V_{3T} + [U_3, V_3] = 0$. This proves the complete integrability of Eq. (33) which in turn proves the complete integrability of the HNLS fiber system for the simultaneous propagation of $N$ nonlinear fields in the normal dispersion regime with the conditions (6).

In order to construct dark soliton solutions using the Hirota method, we apply the following form of bilinear transformation to Eq. (33):

$$u_j = \frac{g_j(Z,T)}{f(Z,T)}, \tag{37}$$

where $g_j(Z,T)$ are complex functions with respect to $Z$ and $T$. Using Eq. (37), Eq. (33) can be decoupled into

$$(iD_Z - D_T^3 + 3\lambda_3 D_T)(g_j f) = 0, \tag{38a}$$

$$(D_T^2 - \lambda_3) ff = -4 \sum_{n=1}^{N} |g_n|^2, \tag{38b}$$

$$D_T(g_j g_{j+1}) = 0 \quad (j = 1, \ldots, N-1), \tag{38c}$$

$$D_T(g_N g_1) = D_T(g_j g_j^*) = 0. \tag{38d}$$

To obtain the dark soliton solutions, we assume

$$g_j = \tau_{3j}(1 + \epsilon g_{1j}), \quad f = 1 + \epsilon f_1, \tag{39}$$

where $\tau_{3j}$ are complex constants and $g_{1j}$ are complex functions of $Z$ and $T$. Substituting Eq. (39) into Eq. (38) and collecting the coefficients of different powers of $\epsilon$, we derive the solutions

$$g_j = -f_1 = -\exp[m_3(T - \lambda_3 Z) + \xi_3^{(0)}], \tag{40}$$

where $\lambda_3 = m_3^2/2 = 4\Sigma_{n=1}^{N} |\tau_{3n}|^2$ and $\xi_3^{(0)}$ is a real constant. Now, using the solutions of $g_{1j}, f_1$, and $\lambda_3$ in Eq. (39) and then in Eq. (37), the dark soliton solutions can be derived as

$$u_j = \tau_{3j} \exp(\pm i\pi) \tanh\left\{ \frac{1}{2}\left[ m_3\left( T - \frac{m_3^2}{2} Z \right) + \xi_3^{(0)} \right] \right\}. \tag{41}$$

### V. DISCUSSION AND CONCLUSIONS

In [11,20] the dark soliton solutions for two coupled CNLS and CH equations were constructed using the Hirota bilinear method. Here, we have derived the same types of dark soliton solutions for $N$ coupled equations. For the HNLS case, in [25], similar dark soliton solutions to Eq. (41) were derived for two coupled HNLS equations. Those authors claimed that for a particular choice of parametric conditions there is the possibility of both dark and bright solitons propagating in both the normal and anomalous dispersion regimes of the optical fiber system. In that work, to derive the soliton solutions from the Hirota bilinear method the authors used only one condition, $k''\gamma = \beta k'''$. Then with the freedom available with the parameter $\gamma_s$, they had the possibility of suggesting that both types of soliton can propagate in both types of dispersion regime. We agree that their argument is correct from the availability of soliton solutions from the Hirota bilinear method with only one condition $k''\gamma = \beta k'''$. But from the complete integrability conditions (6) both from Painlevé analysis [26] and from the above construction of the Lax pair, we find that another condition $\gamma = -2\gamma_s$ must also be satisfied. This condition makes it clear that only a dark soliton will propagate in the normal dispersion regime and only a bright soliton will propagate in the anomalous dispersion regime even in the presence of higher order effects.

To conclude, we have considered the $N$ CNLS, $N$ CH, and

$N$ CHNLS equations which describe simultaneous $N$ nonlinear wave propagation in a fiber medium with important higher order effects. The linear eigenvalue problem associated with the integrable form of the $N$ CNLS, $N$ CH, and $N$ CHNLS equations for the normal dispersion regime was constructed using the AKNS method and the exact form of the dark soliton solutions was also derived using the Hirota bilinear method. Finally, we have shown that only dark (bright) solitons will propagate in the normal (anomalous) dispersion regime of the fiber system even in the presence of higher order effects.

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